Quadratic Expressions & Equations

QUADRATIC EXPRESSION: If $a \neq 0, b, c$ are complex numbers then $ax^2 + bx + c$ is called a quadratic expression in x.

QUADRATIC EQUATION: If $a \neq 0, b, c$ are complex numbers then $ax^2 + bx + c = 0$ is called a quadratic equation in x.

ROOT OF A QUADRATIC EQUATION: If $a\alpha^2 + b\alpha + c = 0$ then α is a root or solution of the quadratic equation $ax^2 + bx + c = 0$.

A quadratic equation can not have more than two roots or two solutions. The roots of $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and its discrminent is $\triangle = b^2 - 4ac$.

NATURE OF THE ROOTS OF THE EQUATION $ax^2 + bx + c = 0$

1.If a,b,c are real and $\triangle > 0$, then the roots are real and distinct.

2.If a,b,c are real and $\triangle = 0$, then the roots are real and equal.

3.If a,b,c are real and $\triangle < 0$, then the roots are two conjugate complex numbers.

4.If a,b,c are rational and $\triangle > 0$, and is a perfect square then the roots are rational and distinct.

5.If a,b,c are rational and $\triangle > 0$, and is not a perfect square then the roots are conjugate surds i.e $\alpha \pm \beta$.

6.If a,b,c are rational and $\triangle < 0$, then the roots are conjugate complex numbers i.e, $\alpha \pm i\beta$.

FORMATION OF THE QUADRATIC EQUATION WITH ROOTS α AND β: The quadratic equation whose roots are α and β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0 \Rightarrow (x - \alpha)(x - \beta) = 0$.

RELATION BETWEEN THE ROOTS α, β OF $ax^2 + bx + c = 0$.

1.$\alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$.

2.$|\alpha - \beta| = \frac{\sqrt{b^2-4ac}}{|a|}$.

3.$\alpha^2 + \beta^2 = \frac{b^2-2ac}{a^2}$.

4.$\alpha^3 + \beta^3 = \frac{3abc-b^3}{a^3}$.

5.$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{-b}{c}$.

6.$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{b^2-2ac}{c^2}$.

7.$\frac{1}{(a\alpha+b)} + \frac{1}{(a\beta+b)} = \frac{b}{ac}$.

8.$\frac{1}{(a\alpha+b)^2} + \frac{1}{(a\beta+b)^2} = \frac{b^2-2ac}{a^2c^2}$.

9.$\frac{1}{(a\alpha+b)^3} + \frac{1}{(a\beta+b)^3} = \frac{b^3-3abc}{a^3c^3}$.

10.$|\alpha^2 - \beta^2| = \frac{|b|\sqrt{b^2-4ac}}{a^2}$.

PROPERTIES OF ROOTS OF THE EQUATION $ax^2 + bx + c = 0.$

If a and c are of the same sign i.e, $\frac{c}{a}$is +ve, then both the roots are of same sign.

If a and c are of opposite sign i.e, $\frac{c}{a}$is -ve, then the roots are of opposite sign.

If both the roots are -ve, then a,b,c will have the same sign.

If both the roots are +ve then a, c will have the same sign different from the sign of b.

If a=c, then the roots are reciprocal to each other.

If a+b+c=0,then the roots are 1 and $\frac{c}{a}$.

If a+c=b, then the roots are -1 and $\frac{-c}{a}$.

If the roots are in the ratio m:n then $(m+n)^2ac = mnb^2$.

If one root is p times the other root then $(1+p)^2ac = pb^2$.

If one root is equal to the n th power of the other root then $(ac^n)^{\frac{1}{n+1}} + (a^nc)^{\frac{1}{n+1}} + b = 0.$

If one root is square of the other, then $a^2c + ac^2 = b(3ac - b^2)$.

If roots differ by unity, then $b^2 = 4ac + a^2$.

SAME ROOTS: If $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have the same roots then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

ONE ROOT IS COMMON: The equations $a_1x^2 + b_1x + c_1 + 0$ and $a_2x^2 + b_2x + c_2 = 0$ where $a_1b_2 - a_2b_1 \neq 0, a_1, a_2 \neq 0$, have one common root then $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$ and the common root is $\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$.

SIGNS OF $'a'$ AND $ax^2 + bx + c$:

If the equation $ax^2 + bx + c = 0$ has complex roots $(\triangle < 0)$ then a and $ax^2 + bx + c$ will have same sign $\forall x \in R$.

If the equation $ax^2 + bx + c = 0$ has equal roots then a and $ax^2 + bx + c$ will have same sign $\forall x \in R - [\frac{-b}{2a}]$.

If the equation $ax^2 + bx + c = 0$ has real roots $\alpha, \beta(\triangle > 0, \alpha < \beta)$ then

1.$\alpha < x < \beta \Leftrightarrow a$ and $ax^2 + bx + c$ will have opposite sign.

2.$x < \alpha$ or $x > \beta \Leftrightarrow a$ and $ax^2 + bx + c$ will have same sign.

MAXIMUM OR MINIMUM VALUE O QUADRATIC EXPRESSION

If a > 0, then the minimum value of $ax^2 + bx + c$ is $\frac{4ac - b^2}{4a}$.(This value is attained at $x = \frac{-b}{2a}$).

If a < 0, then the maximum value of $ax^2 + bx + c$ is $\frac{4ac - b^2}{4a}$.(This value is attained at $x = \frac{-b}{2a}$).

If α, β are the roots of $f(x) \equiv ax^2 + bx + c = 0$ then the equation whose roots are

1. $-\alpha, -\beta \quad is \quad f(-x) = 0.$

2. $\frac{1}{\alpha}, \frac{1}{\beta} \quad is \quad f(\frac{1}{x}) = 0.$

3. $k\alpha, k\beta \quad (k \neq 0) \quad is \quad f(\frac{x}{k}) = 0.$

4. $\alpha + k, \beta + k \quad is \quad f(x - k) = 0.$

5. $\alpha^2, \beta^2 \quad is \quad f(\sqrt{x}) = 0.$

6. $\alpha^k, \beta^k \quad is \quad f(\sqrt[k]{x}) = 0.$

7. $\frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta} \quad is \quad f(\frac{x}{1-x}) = 0.$

LOCATING THE ROOTS OF QUADRATIC EQUATION UNDER GIVEN CONDITIONS

Both the roots of equation $ax^2 + bx + c = 0$are greater than a given number $'k'$if

$Tex - Error2. \quad af(k) > 0.3. \quad \frac{-b}{2a} > k.$

Both the roots of equation $ax^2 + bx + c = 0$are smaller than a given number $'k'$if

$Tex - Error2. \quad af(k) > 0.3. \quad \frac{-b}{2a} < k.$

Exactly one root of $ax^2 + bx + c = 0$lies between the numbers $'p' \quad and \quad 'q'$if

$f(p)f(q) \leq 0$but f(p) and f(q) are not simultaneously zero.

Both the roots of equation $ax^2 + bx + c = 0$lie between two given numbers $'p' \quad and \quad 'q' \quad (p < q)$if

$Tex - Error2. \quad af(p) > 0.3. \quad af(q) > 0.4. \quad p < \frac{-b}{2a} < q.$

The extreme values of $f(x) = \frac{x^2 - ax + b}{x^2 + ax + b}$are $f(\sqrt{-b}), f(\sqrt{b}).$
